# A note on potentially $K_4 - e$ graphical sequences *

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**Abstract**

A sequence $S$ is potentially $K_4 - e$ graphical if it has a realization containing a $K_4 - e$ as a subgraph. Let $\sigma(K_4 - e, n)$ denote the smallest degree sum such that every $n$-term graphical sequence $S$ with $\sigma(S) \geq \sigma(K_4 - e, n)$ is potentially $K_4 - e$ graphical. Gould, Jacobson, Lehel raised the problem of determining the value of $\sigma(K_4 - e, n)$. In this paper, we prove that $\sigma(K_4 - e, n) = 2[(3n - 1)/2]$ for $n \geq 7$, and $n = 4, 5$, and $\sigma(K_4 - e, 6) = 20$.

## 1.Introduction

If $S = (d_1, d_2, ..., d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph $G$ of order $n$, whose degree sequence $(d(v_1), d(v_2), ..., d(v_n))$ is precisely $S$. If $G$ is such a graph then $G$ is said to realize $S$ or be a realization of $S$. A graphical sequence $S$ is potentially $H$ graphical if there is a realization of $S$ containing $H$ as a subgarph, while $S$ is forcibly $H$ graphical if every realization of $S$ contains $H$ as a subgraph. Let $\sigma(S) = d_1 + d_2 + ... + d_n$, $[x]$ be the largest integer less than or equal to $x$, If $G$ and $G_1$ are graphs, then $G \cup G_1$ is the disjoint union of $G$ and $G_1$. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. Let $K_k$ be a complete graph on $k$ vertices, $C_k$ be a cycle of length $k$.

Given a graph $H$, what is $ex(n, H)$, the maximum number of edges of a graph with $n$ vertices not containing $H$ as a subgraph? This problem was proposed for $H = C_4$ by Erdos [2] in 1938 and in general by Turan [9]. In the terms of graphic sequences, the number $2ex(n, H)+2$ is the minimum even integer $m$ such that every $n$-term graphical sequence $S$ with $\sigma(S) \geq m$ is forcibly $H$ graphical. Here we consider the following variant: determine the minimum even integer $m$ such that every $n$-term graphical

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sequence $S$ with $\sigma(S) \geq m$ is potentially $H$ graphical, We denote this minimum $m$ by $\sigma(H, n)$. Erdos, Jacobson and Lehel [1] show that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$; and conjecture that $\sigma(K_k, n) = (k-2)(2n-k+1)+2$; They proved that if $S$ does not contain zero terms, this conjecture is true for $k = 3, n \geq 6$, Li and Song [6,7,8] proved that if $S$ does not contain zero terms, this conjecture is true for $k = 4, n \geq 8$ and $k = 5, n \geq 10$, and $\sigma(K_k, n) \leq 2n(k-2)+2$ for $n \geq 2k-1$. Gould, Jacobson and Lehel [3] proved that this conjecture is true for $k = 4, n \geq 9$, if $n = 8$ and $\sigma(S) \geq 28$, then either there is a realization of $S$ containing $K_4$ or $S = (4^7, 0^1)$(i.e. $S$ consists of 7 integers 4 and 1 integer 0); $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2; \sigma(C_4, n) = 2[(3n-1)/2]$ for $n \geq 4, \sigma(C_4, n) \leq \sigma(K_4-e, n) \leq \sigma(K_4, n)$; and raised the problem of determining the value of $\sigma(K_4-e, n)$. Lai[4,5] proved that $\sigma(C_{2m+1}, n) = m(2n-m-1)+2$, for $m \geq 2, n \geq 3m$; $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$, for $m \geq 2, n \geq 5m-2$. In this paper, we determine the values of $\sigma(K_4-e, n)$.

## 2.$\sigma(K_4 - e, n)$

**Theorem 1.** For $n = 4, 5$ and $n \geq 7$

$$\sigma(K_4-e, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

For $n = 6$, $S$ is a 6-term graphical sequence with $\sigma(S) \geq 16$, then either there is a realization of S containing $K_4 - e$ or $S = (3^6)$. (Thus $\sigma(K_4 - e, 6) = 20$).

Proof. By [3],

$$\sigma(K_4-e, n) \geq \sigma(C_4, n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd} \\ 3n-2 & \text{if } n \text{ is even.} \end{cases}$$

For $n \geq 4$. Assume $d_1 \geq d_2 \geq ... \geq d_n \geq 0$.

For $n = 4$, if a graph has size $q \geq 5$, then clearly it contains a $K_4 - e$, so that $\sigma(K_4 - e, n) \leq 3n - 2$.

For $n = 5$, we have $q \geq 7$. There are exactly 4 graphs of order 5 and size 7 and each contains a $K_4 - e$. Thus $\sigma(K_4 - e, n) \leq 3n - 1$.

Suppose for $5 \leq t < n, S_1$ is a $t$-term graphical sequence such that

$$\sigma(S_1) \geq \begin{cases} 3t-1 & \text{if } t \text{ is odd} \\ 3t-2 & \text{if } t \text{ is even.} \end{cases}$$

Then either $S_1$ has a realization containing a $K_4 - e$ or $S_1 = (3^6)$.

If $n$ is even, $S$ is a $n$-term graphical sequence, $\sigma(S) \geq 3n-2$. Let $G$ be a realization of $S$. Assume $d_1 \geq d_2 \geq ... \geq d_n \geq 0$.

Case 1: Suppose $\sigma(S) = 3n-2$. If $d_n \leq 1$, let $S'$ be the degree sequence of $G - v_n$. Then $\sigma(S') \geq 3n-2-2 = 3(n-1)-1$. By induction, $S'$ has a realization containing a $K_4 - e$. Therefore $S$ has a realization containing a $K_4 - e$. Hence, we may assume that $d_n \geq 2$. Since $\sigma(S) = 3n - 2$, then $d_n = d_{n-1} = 2$. Let $v_n$ be adjacent to $x$ and $y$.

If $x$ or $y = v_{n-1}$, let $S$" be the degree sequence of $G - v_n - v_{n-1}$, then $\sigma(S") = 3n - 2 - 6 = 3(n-2) - 2$. Clearly $S" \neq (3^6)$. By induction, $S$" has a realization containing a $K_4 - e$. Hence, $S$ has a realization containing a $K_4 - e$.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, $v_{n-1}$ is adjacent to $x$ and $y$. We now assume that $x$ is adjacent to $y$. Then $G$ contains a $K_4 - e$. Hence, we may assume that $x$ is not adjacent to $y$. Then the edge interchange that removes the edges $xv_{n-1}$ and $yv_n$ and inserts the edges $xy$, $v_nv_{n-1}$ produces a realization $G'$ of $S$ containing $v_{n-1}v_n$, and we are done as before.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, $v_{n-1}$ is not adjacent to $x$. Let $v_{n-1}$ be adjacent to $z_1$ and $z_2$. We first consider the case $x$ is not adjacent to $z_1$. Then the edge interchange that removes the edges $v_{n-1}z_1$ and $v_nx$ and inserts the edges $xz_1$ and $v_{n-1}v_n$ produces a realization $G'$ of $S$ containing $v_{n-1}v_n$. We have reduced this case to a graph $G'$ as above. Next, if $x$ is not adjacent to $z_2$. Similar to previous case, we can prove that $S$ has a realization containing a $K_4 - e$. Finally, if $x$ is adjacent to $z_1$ and $z_2$. We now assume that $z_1$ is adjacent to $z_2$. Then $G$ contains a $K_4 - e$. Hence, we may assume that $z_1$ is not adjacent to $z_2$. Then the edge interchange that removes the edges $v_{n-1}z_1$, $v_{n-1}z_2$ and $v_nx$ and inserts the edges $v_{n-1}v_n$, $z_1z_2$ and $v_{n-1}x$ produces a realization $G'$ of $S$ containing $v_{n-1}v_n$, and we are done as before.

If $x \neq v_{n-1}$ and $y \neq v_{n-1}$, and $v_{n-1}$ is not adjacent to $y$. Similar to previous case, we can prove that $S$ has a realization containing a $K_4 - e$.

Case 2: Suppose $\sigma(S) = 3n$. If $d_n \leq 2$. Let $S'$ be degree sequence of $G - v_n$, then $\sigma(S') \geq 3n - 4 = 3(n-1) - 1$. By induction, $S'$ has a realization containing a $K_4 - e$. Hence, $S$ has a realization containing a $K_4 - e$. Thus, we may assume that $d_n \geq 3$. Then $S = (3^n)$. If $n = 6$. Let $G_1$ be a realization of $(3^6)$. Clearly $G_1$ does not contain a $K_4 - e$. Next, if $n = 4p$ $(p \geq 2)$. Then $pK_4$ is a realization of $S = (3^n)$ which contains a $K_4 - e$. Finally, suppose that $n = 4p + 2$ $(p \geq 2)$. Then $G_1 \cup (p-1)K_4$ is a realization of $S = (3^n)$ which contains a $K_4 - e$.

Case 3: Suppose $3n + 2 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let $S'$ be a degree sequence of $G - v_n$, then $\sigma(S') \geq 3n + 2 - 6 = 3(n-1) - 1$. By induction, $S'$ has a realization containing a $K_4 - e$. Hence, $S$ has a realization containing a $K_4 - e$.

Case 4: Suppose $\sigma(S) \geq 4n$. If $n \geq 8$. By [3] proposition 2 and theorem 4, $S$ has a realization containing a $K_4$. Next, if $n = 6$ and if $4n \leq \sigma(S) \leq 5n - 2$. Then $d_n \leq 4$. Let $S'$ be a degree sequence of $G - v_n$, then $\sigma(S') \geq 4n - 8 = 16 = 3(n-1) + 1$. By induction, $S'$ has a realization containing a $K_4 - e$. Hence, $S$ has a realization containing a $K_4 - e$. Finally, Suppose that $\sigma(S) \geq 5n = 30$. Then $\sigma(S) = 30$. The realization of $S$ is $K_6$ which contains $K_4 - e$.

If $n$ is odd, $S$ is a $n$-term graphical sequence, $\sigma(S) \geq 3n - 1$. Let $G$ be a realization of $S$.

Case 1: Suppose $\sigma(S) = 3n - 1$. Then $d_n \leq 2$. Let $S'$ be degree sequence of $G - v_n$, then $\sigma(S') \geq 3n - 1 - 4 = 3(n-1) - 2$. By induction, either $S'$ has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore $S$ has a realization containing a $K_4 - e$ or $S = (4^1, 3^5, 1^1)$. Clearly, $(4^1, 3^5, 1^1)$ has a realization containing a $K_4 - e$ (see Appendix Figure 1). Hence, $S$ has a realization containing a $K_4 - e$.

Case 2: Suppose $3n + 1 \leq \sigma(S) \leq 4n - 2$. Then $d_n \leq 3$. Let $S'$ be degree sequence of $G - v_n$, then $\sigma(S') \geq 3n + 1 - 6 = 3(n-1) - 2$. By induction, either $S'$ has a realization containing a $K_4 - e$ or $S' = (3^6)$. Therefore $S$ has a realization containing

a $K_4 - e$ or $S = (4^2, 3^4, 2^1), S = (4^3, 3^4)$. Clearly, $(4^2, 3^4, 2^1)$ and $(4^3, 3^4)$ both have a realization containing a $K_4 - e$ (see Appendix Figure 2). Hence, $S$ has a realization containing a $K_4 - e$.

Case 3: Suppose $\sigma(S) \geq 4n$. If $n \geq 9$, then by theorem 4 of [3], $S$ has a realization containing a $K_4$. Next, if $n = 7$ and if $4n \leq \sigma(S) \leq 5n - 1$, then $d_n \leq 4$. Let $S'$ be a degree sequence of $G - v_n$, then $\sigma(S') \geq 4n - 8 = 3n - 1 = 3(n-1) + 2$. Clearly $S' \neq (3^6)$, so by induction, $S'$ has a realization containing $K_4 - e$. Thus $S$ has a realization containing a $K_4 - e$ . Finally, Suppose $\sigma(S) \geq 5n + 1 = 36$. Clearly, $(6^6, 0^1)$ is not graphical. Hence $d_7 \geq 1$ and by theorem 2.2 of [6], $S$ has a realization containing a $K_4$.

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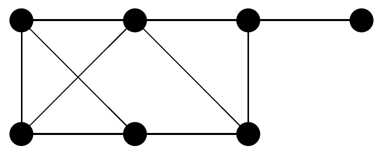
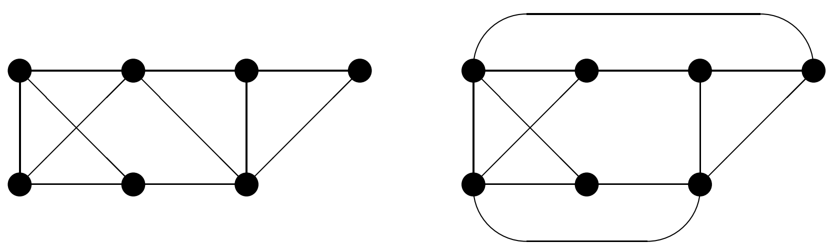
# Appendix



$(4^1, 3^5, 1^1)$

Figure 1



$(4^2, 3^4, 2^1)$ $(4^3, 3^4)$

Figure 2